

Conservation of Angular Momentum in a Flux Qubit

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Oscillations of superconducting current between clockwise and counterclockwise directions in a flux qubit do not conserve the angular momentum of the qubit. To compensate for this effect the solid containing the qubit must oscillate in unison with the current. This requires entanglement of quantum states of the qubit with quantum states of a macroscopic body. The question then arises whether slow decoherence of quantum oscillations of the current is consistent with fast decoherence of quantum states of a macroscopic solid. This problem is analyzed within an exactly solvable quantum model of a qubit embedded in an absolutely rigid solid and for the elastic model that conserves the total angular momentum. We show that while the quantum state of a flux qubit is, in general, a mixture of a large number of rotational states, slow decoherence is permitted if the system is macroscopically large. Practical implications of entanglement of qubit states with mechanical rotations are discussed.

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I. INTRODUCTION

Flux qubits are formed by quantum superposition of current states in a superconducting loop interrupted by one or more Josephson junctions^{1,2}. Quantum mechanics of such a qubit is described by a double-well potential, similar to the textbook example³ of the ammonia molecule, NH_3 . In the latter example the tunneling between states corresponding to the N-atom located to the left (L) or to the right (R) of the H_3 triangle creates quantum superposition of the $|L\rangle$ and $|R\rangle$ states, with the lowest energy doublet given by $|L\rangle \pm |R\rangle$. If one prepares the molecule in, e.g., the $|L\rangle$ state, the quantum mechanical average of the position of the N-atom oscillates harmonically between left and right at the frequency $\omega = \Delta/\hbar$, where Δ is the energy splitting of the doublet. In the rigorous formulation of this problem the N-atom and the H_3 triangle co-tunnel in such a manner that the position of the center of mass of the four atoms is preserved, thus conserving the linear momentum.

In the simplest formulation of the flux qubit problem the role of left and right is played by clockwise and counterclockwise directions of the current. Typical values of the angular momentum associated with the current range from a few hundred \hbar for a submicron SQUID loop⁴, to $10^5 \hbar$ for a micron-size loop⁵, to $10^{10} \hbar$ for larger SQUIDs⁶. To conserve the angular momentum the tunneling of the current between clockwise and counterclockwise directions must be accompanied by quantum transitions between mechanical clockwise and counterclockwise rotations of the body containing the flux qubit. This creates a controversy⁷. Indeed, the co-tunneling of the superconducting current and mechanical rotation needed to conserve the angular momentum requires entanglement of quantum states of the flux qubit with quantum states of a macroscopic body. In any reasonable experiment the phase of the wave function of the equipment containing the flux qubit must be destroyed instantaneously. Then

how can the flux qubit preserve coherence on a measurable time scale? This paper is devoted to the detailed analysis of the entanglement of current states with mechanical rotations and its implications for superconducting qubits.

Within an exactly solvable model of a flux qubit embedded in an absolutely rigid rotator we obtain entangled eigenstates of the system and their dependence on the total angular momentum J . When the system is prepared in the state with a certain direction of the superconducting current, this state is, in general, a quantum mixture of many rotational states of the body. However, only tunnel splittings Δ_J of the states belonging to the same J contribute to the oscillations of the superconducting current. We show that decoherence resulting from the broad statistical distribution over J is small as long as the body containing the qubit is macroscopically large. Thus, contrary to what one might think, the macroscopicity of the body that is entangled with the qubit, is in fact required for low decoherence. We then study decoherence of a flux qubit due to torques generated by the oscillating current in the elastic solid and show how decoherence rates obtained within the two models match. Among other problems we discuss renormalization of the tunnel splitting by the elastic environment and superradiant relaxation in a system of closely packed qubits.

The paper is structured as follows. Exactly solvable quantum model of a flux qubit interacting with rotations of a rigid body is studied in II. Quantum states of the qubit entangled with rotations of the body are obtained in Section II A. Section II B is devoted to decoherence due to rotational excitations of the body. Elastic environment is considered in Section III. The model that conserves the total angular momentum is formulated in Section III A. Section III B discusses decoherence of the flux qubit by internal torques. Renormalization of the tunnel splitting by the elastic environment is computed in Section III C. Section IV contains numerical estimates,

discussion of various effects originating from conservation of angular momentum, alternative interpretations of the results, and final conclusions.

II. RIGID BODY

A. Rotational states of a flux qubit

First, we consider the tunnel-split states of a flux qubit and ignore conservation of the angular momentum. Let the lowest-energy doublet of a flux qubit be

$$\Psi_{\pm} = \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle), \quad (1)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the operator of the angular momentum of the electronic current inside superconducting loop \hat{l}_z ,

$$\begin{aligned} \hat{l}_z |\uparrow\rangle &= l |\uparrow\rangle \\ \hat{l}_z |\downarrow\rangle &= -l |\downarrow\rangle \end{aligned} \quad (2)$$

Eigenfunctions Ψ_{\pm} satisfy

$$\hat{H}\Psi_{\pm} = E_{\pm}\Psi_{\pm} \quad (3)$$

with \hat{H} being the Hamiltonian of the qubit and

$$E_- - E_+ \equiv \Delta \quad (4)$$

being the tunnel splitting. It is convenient to describe such a two-state system by a pseudospin 1/2. Components of the corresponding Pauli operator σ are

$$\begin{aligned} \sigma_x &= |\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| \\ \sigma_y &= i|\downarrow\rangle\langle\uparrow| - i|\uparrow\rangle\langle\downarrow| \\ \sigma_z &= |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|. \end{aligned} \quad (5)$$

The projection of \hat{H} onto $|\uparrow\rangle$ and $|\downarrow\rangle$ states is

$$\hat{H}_{\sigma} = \sum_{m,n=\uparrow,\downarrow} \langle m|\hat{H}|n\rangle |m\rangle\langle n|. \quad (6)$$

According to Eq. (1),

$$\begin{aligned} |\uparrow\rangle &= \frac{1}{\sqrt{2}} (\Psi_+ + \Psi_-) \\ |\downarrow\rangle &= \frac{1}{\sqrt{2}} (\Psi_+ - \Psi_-). \end{aligned} \quad (7)$$

It is now easy to see from Eq. (3) that

$$\begin{aligned} \langle\uparrow|\hat{H}|\uparrow\rangle &= \langle\downarrow|\hat{H}|\downarrow\rangle = 0 \\ \langle\downarrow|\hat{H}|\uparrow\rangle &= \langle\uparrow|\hat{H}|\downarrow\rangle = -\Delta/2. \end{aligned} \quad (8)$$

With the help of these relations one obtains from equations (5) and (6)

$$\hat{H}_{\sigma} = -(\Delta/2)\sigma_x. \quad (9)$$

The general form of the wave function of our two-state system is

$$\Psi(t) = C_+ \Psi_+ e^{i\Delta t/(2\hbar)} + C_- \Psi_- e^{-i\Delta t/(2\hbar)} \quad (10)$$

with $|C_-|^2 + |C_+|^2 = 1$. If one imposes the initial condition $\Psi(0) = |\uparrow\rangle$, then

$$\Psi(t) = \cos\left(\frac{\Delta t}{2\hbar}\right) |\uparrow\rangle + \sin\left(\frac{\Delta t}{2\hbar}\right) |\downarrow\rangle \quad (11)$$

and $\langle\hat{l}_z\rangle = l\langle\sigma_z\rangle$, with

$$\langle\sigma_z\rangle = \langle\Psi(t)|\sigma_z|\Psi(t)\rangle = \cos\left(\frac{\Delta t}{\hbar}\right). \quad (12)$$

This equation describes harmonic oscillations of the superconducting current at the frequency Δ/\hbar between clockwise and counterclockwise directions. Another way to obtain this result is to use the equivalence⁸ of the Schrödinger equation for spin one-half to the precession equation for the expectations value of σ ,

$$\hbar \frac{d}{dt} \left\langle \frac{\sigma}{2} \right\rangle = - \left\langle \sigma \times \frac{\delta \hat{H}_{\sigma}}{\delta \sigma} \right\rangle = \frac{\Delta}{2} \langle \sigma \rangle \times \mathbf{e}_x, \quad (13)$$

which gives

$$\begin{aligned} \frac{d}{dt} \langle \sigma_x \rangle &= 0 \\ \frac{d}{dt} \langle \sigma_y \rangle &= \frac{\Delta}{\hbar} \langle \sigma_z \rangle \\ \frac{d}{dt} \langle \sigma_z \rangle &= -\frac{\Delta}{\hbar} \langle \sigma_y \rangle. \end{aligned} \quad (14)$$

The last two equations give Eq. (12).

We shall account now for mechanical rotations of the body containing the flux qubit. In this Section we shall deal with an absolutely rigid body that can only rotate as a whole. As we shall see, this problem contains all of the components needed to understand the effects of entanglement required by the conservation of the angular momentum. Rotation by the angle ϕ about the quantization axis Z transforms the Hamiltonian of the qubit into

$$\hat{H}' = e^{-i\hat{l}_z\phi} \hat{H} e^{i\hat{l}_z\phi}. \quad (15)$$

Noticing that the operator of the angular momentum of the superconducting current, \hat{l}_z (that is chosen in units of \hbar), commutes with ϕ it is easy to project this Hamiltonian onto $|\uparrow\rangle$ and $|\downarrow\rangle$. Simple calculation yields the following generalization of Eq. (9):

$$\begin{aligned} \hat{H}'_{\sigma} &= \sum_{m,n=\uparrow,\downarrow} \langle m|\hat{H}'|n\rangle |m\rangle\langle n| \\ &= -\frac{\Delta}{2} [e^{-2il\phi}\sigma_+ + e^{2il\phi}\sigma_-] \\ &= -\frac{\Delta}{2} [\cos(2l\phi)\sigma_x + \sin(2l\phi)\sigma_y] \end{aligned} \quad (16)$$

where $\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

To develop a rigorous formulation of the problem let us first assume that the body with the qubit is an isolated system in a pure quantum state described by a single wave function. The full Hamiltonian of the system is

$$\hat{H} = \frac{(\hbar \hat{L}_z)^2}{2I} - \frac{\Delta}{2} [\sigma_x \cos(2l\phi) + \sigma_y \sin(2l\phi)], \quad (17)$$

where $\hat{L}_z = -i(d/d\phi)$ and $I \equiv I_z$ is the moment of inertia of the body for rotation about the quantization axis. It is easy to check that this Hamiltonian commutes with the operator of the total angular momentum,

$$\hat{J}_z = \hat{L}_z + \hat{l}_z = -i\frac{d}{d\phi} + l\sigma_z. \quad (18)$$

Consequently, the eigenstates of (17) must be entangled states of \hat{l}_z and \hat{L}_z that are eigenstates of the total angular momentum \hat{J}_z :

$$|\Psi_{J\pm}\rangle = \frac{C_{J\pm}}{\sqrt{2}} |\uparrow\rangle_l \otimes |J-l\rangle_L \pm \frac{C_{J\mp}}{\sqrt{2}} |\downarrow\rangle_l \otimes |J+l\rangle_L, \quad (19)$$

with $J \equiv J_z$. Simple calculation gives

$$C_{J\pm} = \sqrt{1 \pm \frac{1}{\sqrt{1 + \frac{\Delta^2 J^2}{4(\hbar l)^2 (\hbar J)^2}}}} \quad (20)$$

and

$$E_{J\pm} = \frac{(\hbar l)^2}{2I} + \frac{(\hbar J)^2}{2I} \pm \sqrt{\frac{\Delta^2}{4} + \frac{(\hbar l)^2 (\hbar J)^2}{I^2}} \quad (21)$$

for the energy levels. Here \pm corresponds to \mp in Eq. (19) and $J = 0, \pm 1, \pm 2, \dots$

Alternatively, the same results can be obtained in the coordinate frame attached to the current loop. In this case one starts with the Hamiltonian

$$\hat{H}_r = \frac{(\hbar \hat{L}_z)^2}{2I} - \frac{\Delta}{2} \sigma_x = \frac{(\hbar \hat{J}_z - \hbar \hat{l}_z)^2}{2I} - \frac{\Delta}{2} \sigma_x. \quad (22)$$

Its eigenfunctions are

$$|\Psi_{J\pm}\rangle_r = \frac{1}{\sqrt{2}} (C_{J\pm} |\uparrow\rangle_l \pm C_{J\mp} |\downarrow\rangle_l) \otimes |J\rangle, \quad (23)$$

while eigenvalues are given by Eq. (21). The two coordinate frames are related by unitary transformation.

B. Decoherence from rotations

Any real macroscopic system should have some distribution over J . According to Eq. (21), at large I the energies of the states corresponding to different J can be very close. Consequently, a macroscopically large number of different J -states should contribute to the expectation value of any physical quantity. Since the phases

of such states can differ significantly, the question then arises how the coherence of the flux qubit is influenced by this effect. Rigorous answer to this question is given below.

To study decoherence, one should prepare the system in a state with a certain direction of l_z , e.g. $l_z = +l$, and study how $\langle \hat{l}_z \rangle$ would depend on time. Naturally, the initial state should be obtained by subjecting the system to a strong bias field in the direction of the magnetic moment of the current loop. Adding the term $-\frac{1}{2}W\sigma_z$ to the Hamiltonian, it is easy to work out the energy levels of the biased states:

$$E_{J\pm} = \frac{(\hbar l)^2}{2I} + \frac{(\hbar J)^2}{2I} \pm \sqrt{\frac{\Delta^2}{4} + \left[\frac{W}{2} + \frac{(\hbar l)(\hbar J)}{I} \right]^2}. \quad (24)$$

For a large positive bias the states corresponding to the plus sign in the above equation have too high energies and can be ignored. In this limit the relevant energies, up to a constant, are

$$E_{J-} \equiv E_J = \frac{\hbar^2 (J-l)^2}{2I} = \frac{(\hbar L_z)^2}{2I}, \quad (25)$$

in accordance with the expectation that they must be the energies of the rotational states of the body. To make sure that the system is magnetized in the direction of the field, that is $l_z = +l$, it must be put in contact with a thermal bath at temperature T . This provides thermal distribution over E_J with probabilities given by

$$P_J = \frac{1}{Z} \exp\left(-\frac{E_J}{k_B T}\right), \quad Z = \sum_J \exp\left(-\frac{E_J}{k_B T}\right). \quad (26)$$

If at $t = 0$ the field is removed and the system is isolated from the bath, it will be a mixture of J -states,

$$|\Psi_{Jl}\rangle_0 = |l\rangle \otimes |J-l\rangle, \quad (27)$$

with the probability of each J determined by Eq. (26). Time evolution of each J -state is provided by

$$|\Psi_{Jl}\rangle = \frac{C_{J+}}{\sqrt{2}} |\Psi_{J+}\rangle e^{-iE_{J+}t/\hbar} + \frac{C_{J-}}{\sqrt{2}} |\Psi_{J-}\rangle e^{-iE_{J-}t/\hbar}. \quad (28)$$

Consequently, the time dependence of the expectation value of $\hat{l}_z = l\sigma_z$ is determined by

$$\langle \sigma_z \rangle = \sum_J P_J \langle \Psi_{Jl} | \sigma_z | \Psi_{Jl} \rangle. \quad (29)$$

Using the relations

$$\begin{aligned} \langle \Psi_{J+} | \sigma_z | \Psi_{J+} \rangle &= \frac{1}{2} (C_{J+}^2 - C_{J-}^2) \\ \langle \Psi_{J-} | \sigma_z | \Psi_{J-} \rangle &= \frac{1}{2} (C_{J-}^2 - C_{J+}^2) \\ \langle \Psi_{J-} | \sigma_z | \Psi_{J+} \rangle &= \langle \Psi_{J+} | \sigma_z | \Psi_{J-} \rangle = C_{J+} C_{J-} \end{aligned} \quad (30)$$

one obtains

$$\langle \sigma_z \rangle = \sum_J P_J \left[\frac{\beta_J^2}{1 + \beta_J^2} + \frac{1}{1 + \beta_J^2} \cos \left(\frac{\Delta_J}{\hbar} t \right) \right], \quad (31)$$

where

$$\Delta_J = E_{J+} - E_{J-} = \Delta \sqrt{1 + \beta_J^2}, \quad \beta_J = \frac{2(\hbar l)(\hbar J)}{I \Delta}. \quad (32)$$

Notice that only the energy splitting between states belonging to the same J , separated by Δ_J , contribute to $\langle \sigma_z \rangle$. For a given $J \neq 0$ oscillations of the superconducting current occur between $\langle l_z \rangle = l$ and $\langle l_z \rangle = l(\beta_J^2 - 1)/(\beta_J^2 + 1)$ as compared to the oscillations between $\pm l$ for $J = 0$ ($\beta_J = 0$).

Formally, at $T = 0$, only the non-rotating state with $J = l$ contributes to the sum in Eq. (31), providing

$$\langle \sigma_z \rangle = \frac{\beta_l^2}{1 + \beta_l^2} + \frac{1}{1 + \beta_l^2} \cos \left(\frac{\Delta_l}{\hbar} t \right), \quad (33)$$

where β_l equals β_J at $J = l$. For a macroscopic body with a large moment of inertia $\beta_l \ll 1$, so that the difference between Eq. (12) and Eq. (33) is very small. The absence of decoherence at $T = 0$ is related to the fact that the system is in a pure J -state.

At $T \neq 0$ rotations of a macroscopic body must be distributed over a macroscopically large number of $J \gg l$. Consequently, one can replace $J - l$ in Eq. (25) with J and replace summation in Eqs. (31), (26) by integration over J . This gives $Z = \sqrt{2\pi I k_B T} / \hbar$. Expectation value of σ_z depends on time through $(\Delta/\hbar)t$,

$$\langle \sigma_z \rangle = \frac{1}{\sqrt{\pi} \beta_T} \int_{-\infty}^{+\infty} d\beta_J \exp \left(-\frac{\beta_J^2}{\beta_T^2} \right) \times \left[\frac{\beta_J^2}{1 + \beta_J^2} + \frac{1}{1 + \beta_J^2} \cos \left(\sqrt{1 + \beta_J^2} \frac{\Delta}{\hbar} t \right) \right], \quad (34)$$

and is determined by a single parameter,

$$\beta_T = 2 \sqrt{\beta_l \frac{k_B T}{\Delta}} = 2^{3/2} \frac{\hbar l}{\Delta} \sqrt{\frac{k_B T}{I}}. \quad (35)$$

Note that $\beta_l = 2(\hbar l)^2 / (I \Delta)$ contains a macroscopically large number I in the denominator. This provides

$$\beta_l \ll \beta_T \ll 1 \quad (36)$$

for any reasonable values of l , Δ , and T . Since the main contribution to the integral in Eq. (34) comes from $\beta_J \sim \beta_T \gg \beta_l$, the overwhelming majority of J contributing to the integral satisfy $J \gg l$ in accordance with our assumption.

From Eq. (34) the asymptotic value of $\langle \sigma_z \rangle$ is

$$\sigma_\infty \equiv \lim_{t \rightarrow \infty} \langle \sigma_z \rangle = \frac{1}{2} \beta_T^2 = 2\beta_l \frac{k_B T}{\Delta}. \quad (37)$$

For a macroscopic body it is small due to the smallness of β_l . In this limit the time dependence of the oscillating term in Eq. (34) can be computed exactly:

$$\langle \sigma_z \rangle_t = \text{Re} \left[\frac{e^{i(\Delta/\hbar)t}}{\sqrt{1 - i\sigma_\infty(\Delta/\hbar)t}} \right]. \quad (38)$$

One can see that the amplitude of quantum oscillations is decreasing as $1/\sqrt{\sigma_\infty(\Delta/\hbar)t}$. Thus, the effective decoherence rate due to the entanglement of the flux qubit with rotations of the rigid body is

$$\Gamma_r = \sigma_\infty \frac{\Delta}{\hbar} = 2\beta_l \frac{k_B T}{\hbar} = \frac{4\hbar l^2}{I} \left(\frac{k_B T}{\Delta} \right). \quad (39)$$

Notice that slow, $1/\sqrt{t}$, decay of coherent oscillations given by Eq. (38) is a consequence of the absolute rigidity of the body.

Proportionality of Γ_r to $1/I$ illustrates our point that, contrary to the naive picture that one might have⁷, the entanglement of a flux qubit with rotations of a macroscopic body, dictated by the conservation of angular momentum, does not necessarily result in a strong decoherence. This comes as a consequence of the selection rule: According to Eq. (31) only tunnel splittings, $\Delta_J = E_{J+} - E_{J-}$, of the states (21) belonging to the same J contribute to $\langle \sigma_z \rangle$. For a macroscopic body, all Δ_J are very close, thus providing low decoherence.

III. ELASTIC BODY

A. Flux qubit in the elastic environment

Realistically, the body containing a flux qubit is not absolutely rigid. During half-period of oscillations of the superconducting current the elastic stress generated by the changing angular momentum of the current may only extend as far as half-wavelength, $\lambda/2 = \pi \hbar v_t / \Delta$, of the transverse sound of frequency Δ/\hbar and speed v_t . We shall assume that this distance is greater than the size of the current loop. For, e.g., a micron-size loop this condition would be typically fulfilled for $\Delta/\hbar < 10\text{GHz}$. It allows one to treat the flux qubit as a point source of the elastic stress, without considering interactions of segments of the current loop with the elastic environment.

Now the rotation angle ϕ that appears in the previous section is determined to the elastic twist,⁹

$$\phi = \frac{1}{2} [\nabla \times \mathbf{u}]_z, \quad (40)$$

where \mathbf{u} is the phonon displacement field at the location of the flux qubit $\mathbf{r} = 0$. Conventional quantization of phonons gives

$$\phi = \frac{1}{2} \sqrt{\frac{\hbar}{2\rho V}} \sum_{\mathbf{k}\lambda} \frac{[i\mathbf{k} \times \mathbf{e}_{\mathbf{k}\lambda}]_z}{\sqrt{\omega_{\mathbf{k}\lambda}}} (a_{\mathbf{k}\lambda} + a_{-\mathbf{k}\lambda}^\dagger), \quad (41)$$

where $a_{\mathbf{k}\lambda}^\dagger, a_{\mathbf{k}\lambda}$ are operators of creation and annihilation of phonons of wave-vector \mathbf{k} and polarization λ , $\mathbf{e}_{\mathbf{k}\lambda}$ are unit vectors of polarization, $\omega_{\mathbf{k}\lambda} = v_t k$ is the phonon frequency, ρ is the mass density of the solid and V is its volume. Since we limit our consideration to elastic twists, only the two transverse polarizations of sound contribute to Eq. (41).

Elastic Hamiltonian that replaces Hamiltonian (17) of the rigid-body approximation is

$$\begin{aligned} \hat{H} = & \sum_{\mathbf{k}\lambda} \hbar \omega_{\mathbf{k}\lambda} \left(a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} + \frac{1}{2} \right) - \\ & - \frac{\Delta}{2} \left\{ \sigma_+ \exp \left[l \sum_{\mathbf{k}\lambda} \xi_{\mathbf{k}\lambda} \left(a_{\mathbf{k}\lambda} - a_{\mathbf{k}\lambda}^\dagger \right) \right] \right. \\ & + \left. \sigma_- \exp \left[-l \sum_{\mathbf{k}\lambda} \xi_{\mathbf{k}\lambda} \left(a_{\mathbf{k}\lambda} - a_{\mathbf{k}\lambda}^\dagger \right) \right] \right\}, \quad (42) \end{aligned}$$

where

$$\xi_{\mathbf{k}\lambda} \equiv \sqrt{\frac{\hbar}{2\rho V}} \frac{[\mathbf{k} \times \mathbf{e}_{\mathbf{k}\lambda}]_z}{\sqrt{\omega_{\mathbf{k}\lambda}}}. \quad (43)$$

Validity of this approximation relies on the fact that angular velocity of the local rotation, $\Omega = d\phi/dt$, is always small compared to the frequency of sound ω . Indeed, noticing that according to Eq. (40) $\Omega \sim \omega k u$ we see that $\Omega \ll \omega$ coincides with the condition of validity of the elastic theory: $ku \ll 1$.

Unitary transformation $\hat{H}_r = \hat{U}^{-1} \hat{H} \hat{U}$ with

$$\hat{U} = \exp \left[\frac{1}{2} l \sigma_z \sum_{\mathbf{k}\lambda} \xi_{\mathbf{k}\lambda} \left(a_{\mathbf{k}\lambda} - a_{\mathbf{k}\lambda}^\dagger \right) \right] \quad (44)$$

transforms Hamiltonian (42) into

$$\begin{aligned} \hat{H}_r = & \hat{U}^{-1} \left[\sum_{\mathbf{k}\lambda} \hbar \omega_{\mathbf{k}\lambda} \left(a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} + \frac{1}{2} \right) \right] \hat{U} - \frac{\Delta}{2} \sigma_x \\ = & \sum_{\mathbf{k}\lambda} \hbar \omega_{\mathbf{k}\lambda} \left[a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} - \frac{l \sigma_z}{2} \xi_{\mathbf{k}\lambda} \left(a_{\mathbf{k}\lambda} + a_{\mathbf{k}\lambda}^\dagger \right) \right] - \frac{\Delta}{2} \sigma_x, \quad (45) \end{aligned}$$

where an insignificant constant has been omitted. In the transition from the first to the second line of Eq. (45) we have used properties of the displacement operator,

$$\hat{D}^{-1}(\alpha) a \hat{D}(\alpha) = a + \alpha, \quad \hat{D}^{-1}(\alpha) a^\dagger \hat{D}(\alpha) = a^\dagger + \alpha^*, \quad (46)$$

with

$$\hat{D}(\alpha_{\mathbf{k}\lambda}) = e^{-\alpha_{\mathbf{k}\lambda}^* a_{\mathbf{k}\lambda} + \alpha_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\dagger}, \quad \alpha_{\mathbf{k}\lambda} = -\frac{1}{2} l \sigma_z \xi_{\mathbf{k}\lambda}. \quad (47)$$

Eq. (45) shows that from mathematical point of view the problem formulated in this Section is a variance of spin-boson problem¹⁰. While some important theorems have been proved for this problem in recent years (see,

e.g., Ref. 11 and references therein), its exact eigenstates are unknown. This prevents us from developing rigorous mathematical approach to decoherence along the lines of the previous Section. From a physical point of view, the attractiveness of our variance of the spin-boson model is in the absence of free parameters. The boson field in our case is the phonon displacement field. Its coupling to the flux qubit (described by spin 1/2) is completely determined by the conservation of total angular momentum. In what follows, we will use an approximation based upon observation that local twists of the elastic solid due to oscillations of the superconducting current in a flux qubit must be very small. Within this approximation we will describe transverse phonons by a classical displacement field $\mathbf{u}(\mathbf{r}, t)$, satisfying $\nabla \cdot \mathbf{u} = 0$.

Expanding Hamiltonian (42) to the lowest power on the elastic twist and replacing operators by their classical expectation values, one obtains

$$H = H_E - \frac{\Delta}{2} \sigma_x - \frac{\Delta}{2} l \sigma_y \int d^3 r \delta(\mathbf{r}) \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right), \quad (48)$$

where H_E is the Hamiltonian of free rotations,

$$H_E = \frac{1}{4} \int d^3 r \rho v_t^2 \left(\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha} \right)^2. \quad (49)$$

The dynamical equation for the displacement field is

$$\rho \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial \sigma_{\alpha\beta}}{\partial r_\beta}, \quad (50)$$

where $\sigma_{\alpha\beta} = \delta H / \delta e_{\alpha\beta}$ is the stress tensor and $e_{\alpha\beta} = \partial u_\alpha / \partial r_\beta$ is the strain tensor. This gives

$$\rho \left(\frac{\partial^2 u_x}{\partial t^2} - v_t^2 \nabla^2 u_x \right) = -\frac{\Delta}{2} l \sigma_y \frac{\partial}{\partial y} \delta(\mathbf{r}) \quad (51)$$

$$\rho \left(\frac{\partial^2 u_y}{\partial t^2} - v_t^2 \nabla^2 u_y \right) = \frac{\Delta}{2} l \sigma_y \frac{\partial}{\partial x} \delta(\mathbf{r}). \quad (52)$$

The above equations should be solved together with the Landau-Lifshitz equation for $\boldsymbol{\sigma}$:

$$\frac{\hbar}{2} \frac{d\boldsymbol{\sigma}}{dt} = -\boldsymbol{\sigma} \times \frac{\delta H}{\delta \boldsymbol{\sigma}}, \quad (53)$$

which gives

$$\hbar \frac{d\sigma_x}{dt} = -\sigma_z \Delta l \int d^3 r \delta(\mathbf{r}) \left[\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right] \quad (54)$$

$$\hbar \frac{d\sigma_y}{dt} = \sigma_z \Delta \quad (55)$$

$$\hbar \frac{d\sigma_z}{dt} = -\sigma_y \Delta + \sigma_x \Delta l \int d^3 r \delta(\mathbf{r}) \left[\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right] \quad (56)$$

It is easy to see that Eqs. (54), (55) and (56) preserve the length of $\boldsymbol{\sigma}$: $\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 1$.

First, let us show that, in accordance with our general line of reasoning, the above equations conserve the Z -component of the total angular momentum,

$$J_z = \hbar l \sigma_z + L_z. \quad (57)$$

Here L_z is the Z -component of the mechanical angular momentum. Its time derivative equals the Z -component of the total mechanical torque, K_z , acting on the body. In the absence of the external torque applied to the surface of the body, K_z is given by⁹

$$K_z = \int d^3r (\sigma_{yx} - \sigma_{xy}). \quad (58)$$

Conventional elastic theory postulates no internal torques, in which case the stress tensor would be symmetric and K_z would be zero. Situation changes when there are transitions between angular momentum states of a microscopic object inside the body, such as, e.g., a flux qubit. In this case the stress tensor is non-symmetric, yielding

$$\frac{dL_z}{dt} = \int d^3r (\sigma'_{yx} - \sigma'_{xy}), \quad (59)$$

where $\sigma'_{\alpha\beta} = \delta H_{\text{int}} / \delta e_{\alpha\beta}$ is the part of the stress tensor related to the interaction of the flux qubit with the elastic environment, H_{int} . The latter is given by the second term in Eq. (42). To prove conservation of the total angular momentum one needs to write this term with the accuracy to second-order terms on the elastic twists:

$$H_{\text{int}} = -\frac{l}{2} \Delta \sigma_y \int d^3r \delta(\mathbf{r}) \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right) + \frac{l^2}{4} \Delta \sigma_x \left[\int d^3r \delta(\mathbf{r}) \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right) \right]^2. \quad (60)$$

This gives

$$\begin{aligned} \sigma'_{xy} &= -\frac{l}{2} \Delta \sigma_y \delta(\mathbf{r}) + \frac{l^2}{2} \Delta \sigma_x \delta(\mathbf{r}) \int d^3r \delta(\mathbf{r}) \left[\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right] \\ \sigma'_{yx} &= \frac{l}{2} \Delta \sigma_y \delta(\mathbf{r}) - \frac{l^2}{2} \Delta \sigma_x \delta(\mathbf{r}) \int d^3r \delta(\mathbf{r}) \left[\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right] \end{aligned} \quad (61)$$

so that

$$\begin{aligned} \frac{dJ_z}{dt} &= \hbar l \frac{d\sigma_z}{dt} + \frac{dL_z}{dt} = \hbar l \frac{d\sigma_z}{dt} + l \Delta \sigma_y \\ &\quad - l^2 \Delta \sigma_x \int d^3r \delta(\mathbf{r}) \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right). \end{aligned} \quad (62)$$

It is now easy to see that condition $dJ_z/dt = 0$ coincides with one of the equations of motion, Eq. (56).

B. Decoherence from internal torques

At $\mathbf{u} = 0$ equations (55) and (56) would describe coherent precession of $\boldsymbol{\sigma}$ about the X -axis, with $\sigma_x = \text{const}$,

$\sigma_z \propto \cos(t\Delta/\hbar)$, and $\sigma_y \propto \sin(t\Delta/\hbar)$. Conservation of angular momentum makes the flux qubit wiggle mechanically when the current oscillates between clockwise and counterclockwise. Consequently, it becomes a source of sound, as can be seen from Eqs. (51) and (52). Let us linearize all equations of motion around $\sigma_x = 1$, $\mathbf{u} = 0$, with small $\sigma_{y,z}(t) \propto e^{-i\omega t}$ and

$$u_{x,y}(\mathbf{r}, t) \propto e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} u_{x,y}(\mathbf{k}). \quad (63)$$

Writing $\delta(\mathbf{r})$ as $\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}}$ one obtains from Eqs. (51) and (52)

$$u_x(\mathbf{k}) = -\frac{l\Delta}{2\rho} \frac{ik_y \sigma_y}{k^2 v_t^2 - \omega^2}, \quad u_y(\mathbf{k}) = \frac{l\Delta}{2\rho} \frac{ik_x \sigma_y}{k^2 v_t^2 - \omega^2}, \quad (64)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. Substitution into Eqs. (55) and (56) results in

$$\hbar^2 \omega^2 = \Delta^2 \left(1 - \frac{l^2 \Delta}{2\rho} \int \frac{d^3k}{(2\pi)^3} \frac{k_x^2 + k_y^2}{k^2 v_t^2 - \omega^2} \right) \quad (65)$$

The integral in this equation should be computed in the complex plane with account of a small imaginary part of ω ,

$$\int \frac{d^3k}{(2\pi)^3} \frac{k_x^2 + k_y^2}{k^2 v_t^2 - \omega^2} = \frac{1}{3\pi^2} \int \frac{k^4 dk}{k^2 v_t^2 - \omega^2} = \frac{i\omega^3}{3\pi v_t^5}. \quad (66)$$

This gives

$$\hbar^2 \omega^2 = \Delta^2 \left(1 - i \frac{l^2 \omega^3}{6\pi \rho v_t^5} \right), \quad (67)$$

that is,

$$\omega = \frac{\Delta}{\hbar} - i\Gamma_0, \quad (68)$$

where

$$\Gamma_0 = \frac{l^2 \Delta^5}{12\pi \hbar^4 \rho v_t^5} \quad (69)$$

is the $T = 0$ rate of the decay of the coherent precession of $\boldsymbol{\sigma}$. This result is in full agreement with the decoherence rate computed with the help of the Fermi golden rule by considering spontaneous quantum transition from the excited state ($|l\rangle - |-l\rangle$) to the ground state ($|l\rangle + |-l\rangle$) with the radiation of a phonon of energy Δ ¹². Its generalization to finite temperature is $\Gamma_e = \Gamma_0 \coth[\Delta/(2k_B T)]$. At $k_B T \gg \Delta$ it gives $\Gamma_e \propto T$ as in Eq. (39) obtained for the rigid body. Comparison of the decoherence provided by the two models will be done in Section IV.

As is clear from the derivation, the above result corresponds to the decoherence of a weakly excited state of the flux qubit. Our method, however, permits study of decoherence of the state prepared with $\mathbf{u} = 0$ and arbitrary σ_z (including $\sigma_z = 1$) at $t = 0$. Dynamics of the vector $\boldsymbol{\sigma}$

consists of fast precession about the X -axis and slow relaxation towards the energy minimum that according to Eq. (48) corresponds to $\sigma_x = 1$, $\sigma_{y,z} = 0$. It is accompanied by radiation of sound due to the torque acting on the flux qubit from the oscillating current. Noticing that the space-time Fourier transform of the displacement generated by the torque, $\mathbf{u}(\mathbf{k}, \omega)$, and the time Fourier transform, $\boldsymbol{\sigma}(\omega)$, of $\boldsymbol{\sigma}(t)$ are always related by Eqs. (64) due to the linearity of Eqs. (51) and (52), one can transform the integral in Eqs. (54) and (56) as

$$\int d^3r \delta(\mathbf{r}) \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right) = \frac{l\Delta}{6\pi v_t^5} \int \frac{d\omega}{2\pi} i\omega^3 \sigma_y(\omega) e^{-i\omega t}. \quad (70)$$

To the first approximation, fast-precessing and slowly-relaxing solution of Eqs. (55) and (56) that satisfies $\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 1$ is

$$\begin{aligned} \sigma_y(t) &= \sqrt{1 - \langle \sigma_x \rangle^2} \sin\left(\frac{\Delta}{\hbar} t\right) \\ \sigma_z(t) &= \sqrt{1 - \langle \sigma_x \rangle^2} \cos\left(\frac{\Delta}{\hbar} t\right), \end{aligned} \quad (71)$$

where $\langle \sigma_x \rangle$ is a slow function of time. Within this approximation the Fourier transform of σ_y in Eq. (70) is dominated by the Fourier transform of $\sin(t\Delta/\hbar)$ that equals

$$i\pi[\delta(\omega + \Delta/\hbar) - \delta(\omega - \Delta/\hbar)], \quad (72)$$

so that the integral (70) becomes

$$-2 \frac{\hbar\Gamma_0}{\Delta} \sqrt{1 - \langle \sigma_x \rangle^2} \cos\left(\frac{\Delta}{\hbar} t\right) \quad (73)$$

where Γ_0 is given by Eq. (69). Substituting this result into Eq. (54), taking into account the first of Eqs. (71), and averaging the resulting equation over fast oscillations, $\langle \cos^2(t\Delta/\hbar) \rangle = 1/2$, one obtains

$$\frac{\partial \langle \sigma_x \rangle}{\partial t} = \Gamma_0 (1 - \langle \sigma_x \rangle^2). \quad (74)$$

This leads to the following relaxation law at $t > 0$ after the system was prepared in the state with arbitrary $\sigma_x = \tanh(\Gamma_0 t_0) \leq 1$ at the moment of time $t = 0$:

$$\langle \sigma_x \rangle = \tanh[\Gamma_0(t + t_0)] \quad (75)$$

$$\sigma_y = \frac{\sin\left(\frac{\Delta}{\hbar} t\right)}{\cosh[\Gamma_0(t + t_0)]} \quad (76)$$

$$\sigma_z = \frac{\cos\left(\frac{\Delta}{\hbar} t\right)}{\cosh[\Gamma_0(t + t_0)]}. \quad (77)$$

Our previous consideration of small oscillations of $\sigma_{y,z}$ (that is, precession around $\sigma_x \rightarrow 1$) corresponds to the choice of $\Gamma_0 t_0 \gg 1$, in which case the decay of the oscillations is always exponential with the rate Γ_0 , as has been previously found. If the system is prepared in the state

with $\sigma_z = 1$ (that corresponds to the choice of $t_0 = 0$ in the above equations), it exhibits exponential relaxation,

$$\sigma_z = 2e^{-\Gamma_0 t} \cos(t\Delta/\hbar), \quad (78)$$

only at $\Gamma_0 t \gg 1$. The initial relaxation at $\Gamma_0 t \ll 1$ is slower:

$$\sigma_z = \frac{\cos(t\Delta/\hbar)}{1 + \frac{1}{2}(\Gamma_0 t)^2}. \quad (79)$$

This later result for a two-state system should be taken with a grain of salt, though, as it is likely to be the consequence of the approximation in which the expectation value of the second term in Eq. (48) is replaced by the product of expectation values of σ_y and phonon field. Such approximation neglects quantum correlations between spin 1/2 and the boson field. In this connection, it is interesting to notice that our model can be easily extended to a system of more than one flux qubit if all the qubits have the same resonance frequency, $\omega = \Delta/\hbar$, and are located within a distance from each other that is small compared to the wavelength of sound of frequency ω . Indeed, for such a system $\boldsymbol{\sigma}/2$ in Eq. (42) gets replaced with the total effective spin $\mathbf{S} = \boldsymbol{\sigma}_1/2 + \boldsymbol{\sigma}_2/2 + \boldsymbol{\sigma}_3/2 + \dots$. Since the resulting Hamiltonian is linear on \mathbf{S} , it commutes with \mathbf{S}^2 . Consequently, when the number of qubits, N , is large, \mathbf{S} must behave as a classical large spin of constant length. In this case, the approximation that neglects quantum correlations must be good. It leads to the same equations (51) - (56) in which $\boldsymbol{\sigma}$ is replaced with $N\boldsymbol{\sigma}$. This amplifies the amplitude of sound by a factor N . Consequently, Γ_0 is amplified by a factor N^2 . One immediately recognizes Dicke superradiance¹³ in this effect. We, therefore, expect Eqs. (75) - (77) with $\Gamma_0 \rightarrow N^2\Gamma_0$ to correctly describe decoherence in a system of $N \gg 1$ closely packed flux qubits.

C. Renormalization of the tunnel splitting by the elastic environment

The above consideration shows that decoherence of the flux qubit in the elastic environment is dominated by phonons of energy Δ . Meantime, even at $T = 0$ there are zero-point oscillations of the solid that produce elastic twists. Such twists interact with the flux qubit and, as we shall see below, renormalize the tunnel splitting. This problem cannot be treated semiclassically as it requires consideration of the entanglement of the qubit with the excitation modes of the solid. It is based upon computation of the quantum average of the Hamiltonian (42), $\langle 0|\hat{H}|0\rangle$ over the ground state of the solid, $|0\rangle$, that has no real phonons.

Noticing that

$$\begin{aligned} \langle 0|e^{l\xi_{\mathbf{k}\lambda}(a_{\mathbf{k}\lambda} - a_{\mathbf{k}\lambda}^\dagger)}|0\rangle &= \langle 0|e^{-l\xi_{\mathbf{k}\lambda}(a_{\mathbf{k}\lambda} - a_{\mathbf{k}\lambda}^\dagger)}|0\rangle \\ &= 1 - \frac{1}{2}|l\xi_{\mathbf{k}\lambda}|^2 + \dots = e^{-|l\xi_{\mathbf{k}\lambda}|^2/2}, \end{aligned} \quad (80)$$

one obtains

$$\hat{H}_\sigma \equiv \langle 0 | \hat{H} | 0 \rangle = -\frac{\Delta}{2} \exp \left(-\frac{l^2}{2} \sum_{\mathbf{k}\lambda} |\xi_{\mathbf{k}\lambda}|^2 \right) (\sigma_+ + \sigma_-), \quad (81)$$

that is,

$$\hat{H}_\sigma = -\frac{\Delta_{\text{eff}}}{2} \sigma_x, \quad (82)$$

where

$$\Delta_{\text{eff}} = \Delta \exp \left(-\frac{l^2}{2} \sum_{\mathbf{k}\lambda} |\xi_{\mathbf{k}\lambda}|^2 \right) \quad (83)$$

is the tunnel splitting renormalized by zero-point quantum elastic twists. Here $\xi_{\mathbf{k}\lambda}$ is given by Eq. (43).

The sum over \mathbf{k} in Eq. (83) can be computed by replacing it with the integral $V \int d^3k / (2\pi)^3$. For the two transverse phonon modes $\mathbf{k} \times \mathbf{e}_{\mathbf{k}t_1} = \pm k \mathbf{e}_{\mathbf{k}t_2}$. Averaging over the angles then gives $\langle |\mathbf{e}_{\mathbf{k}t}|^2 \rangle = 1/3$. Integrating over k from zero to k_{max} determined by the size of the flux qubit, one obtains

$$\Delta_{\text{eff}} = \Delta \exp \left(-\frac{\hbar l^2 k_{\text{max}}^4}{48\pi^2 \rho v_t} \right) \quad (84)$$

A quick estimate (see Section IV) shows that the exponent in Eq. (84) is always small, thus providing negligible renormalization of the tunnel splitting in a flux qubit. However, the above result illustrates an important point. If, for some reason, the shear modulus of the solid, $G = \rho v_t^2$, disappeared, this, according to Eq. (84), would lead to the disappearance of the tunnel splitting as well. The latter is a consequence of the conservation of angular momentum: The current cannot reverse direction if it cannot transfer momentum to the body. As is discussed in the next Section this effect may, in principle, be observed in some two-state systems.

IV. DISCUSSION AND CONCLUSIONS

We have studied two models that take into account mechanical effects associated with quantum oscillations of a superconducting current in a flux qubit. These effects have simple physical origin. To change direction, the current must transfer momentum to the underlying crystal lattice. For the current oscillating in a SQUID loop, it is a microscopic analogue of the Einstein - de Haas effect: The change in the angular momentum of the current associated with its magnetic moment must be compensated by the change in the angular momentum of the body containing the current. This inevitably entangles quantum states of a flux qubit with quantum states of a macroscopic body containing the qubit. One can naively imagine that almost instantaneous decoherence of quantum states of the macroscopic body would have a detrimental effect on the decoherence of the flux

qubit. We show that this is not the case due to the selection rule originating from conservation of angular momentum. While quantum state of a macroscopic system is, in general, an admixture of a large number of rotational states corresponding to different total angular momenta, only tunnel splittings of the states belonging to the same J contribute to quantum oscillations of the superconducting current. Broadening of the tunnel splitting by the rotational states of a qubit is small as long as the body is sufficiently large.

In the first part of the paper we have studied an exactly solvable model of a flux qubit entangled with a rigid mechanical rotator. We show that decoherence in such a system is weak due to inverse proportionality of the decoherence rate, $\Gamma_r = (4\hbar l^2/I)(k_B T/\Delta)$, to the moment of inertia of the rotator, I . To put things in perspective, consider, e.g., a micron-size flux qubit embedded in a body of a comparable small size that is free to rotate. Sound of frequency $\omega = \Delta/\hbar \sim 10^{10} \text{s}^{-1}$ would have a wavelength comparable to the size of the body. Consequently, in reaction to the oscillations of the superconducting current, such a system would rotate as a whole, making the rigid-body approximation developed in Section II a reasonably good one. Typical value of the moment of inertia of a micron-size body is in the ballpark of $10^{-19} \text{g}\cdot\text{cm}^2$. Taking $l \sim 10^5$ for a micron-size current loop, one obtains the following values of the parameters in equations (35) - (39): $\beta_l \approx 2 \times 10^{-8}$, $\beta_T \approx 3 \times 10^{-4} (k_B T/\Delta)^{1/2}$, $\sigma_\infty \sim 4 \times 10^{-8} (k_B T/\Delta)$. Decoherence is dominated by $J \sim 10^9 (k_B T/\Delta)^{1/2}$, which corresponds to frequencies of the rotational Brownian motion $\omega = \hbar J/I \sim 10 (k_B T/\Delta)^{1/2} \text{s}^{-1}$. This provides $\Gamma \sim 500 \text{s}^{-1}$ that corresponds to a rather high quality factor of quantum oscillations, $Q = \Delta/(\hbar \Gamma) \sim 2 \times 10^7 [\Delta/(k_B T)]$, even in the extreme case of a micron size system.

In the second part of the paper we have studied interaction of the flux qubit with the twists of the elastic body, dictated by the conservation of angular momentum. Such model has no free parameters. While its exact quantum states are not known, one can develop a reasonably good approximation in which the internal torque produced inside the body by the oscillating current is treated as a source of elastic shear waves. If the elastic environment is considered to be infinite in space, this is an open system as compared to the closed system that consists of a finite-size rotator with a flux qubit. In the infinite elastic system the shear waves generated by the point source of torque escape to infinity, thus allowing finite decoherence at $T = 0$ as compared to the closed system. The corresponding decoherence rate is given by $\Gamma_e = l^2 \Delta^5 / (12\pi \hbar^4 \rho v_t^5) \coth[\Delta/(2k_B T)]$. At $l \sim 10^5$, $\omega = \Delta/\hbar \sim 10^{10} \text{s}^{-1}$ it is of the order of 10^6s^{-1} , which provides $Q = \Delta/(\hbar \Gamma) \sim 10^4$. This shows that the effect studied in this paper, while allowing weak decoherence, can hardly be ignored in designing flux qubits.

A good check of the validity of the above results can be obtained by comparing decoherence rates obtained within the rigid-rotator model and within the elastic

model. At $k_B T \geq \Delta$ the ratio of the two rates is $\Gamma_e/\Gamma_r = (4\pi^4/3)(I/\rho\lambda^5)$ where $\lambda = 2\pi\hbar v_t/\Delta$ is the wavelength of shear waves of frequency $\omega = \Delta/\hbar$. Noticing that the moment of inertia of a rigid body of radius R is of order ρR^5 , we see that $\Gamma_e/\Gamma_r \sim 1$ at $\lambda \sim 2R$. This agreement between the two models that consider the same effect from two very different angles is quite remarkable.

In our consideration of the conservation of angular momentum, certain effects that may exist in real systems have been left out. Among them are interactions of the flux qubit with magnetic atoms and nuclear spins that can, in principle, absorb some part of the angular momentum of the SQUID. For $l \gg 1$ such processes must be suppressed, however, as they require coherent participation of many magnetic atoms and many nuclear spins. Interaction of the flux qubit with the shear waves of the body must be the primary mechanism of the conservation of angular momentum. Being unavoidable, it imposes a universal upper bound on the quality factor of the qubit.

The effect of rotations on decoherence can also be understood from another angle. At $\phi = \omega t$ that corresponds to the uniform rotation of the flux qubit about the Z -axis the Hamiltonian (16) is equivalent to the Hamiltonian of spin $1/2$ in the effective magnetic field of amplitude $\Delta/(2\mu_B)$ (μ_B being the Bohr magneton) rotating in the XY plane at an angular velocity $\Omega = 2\omega$. Switching to the coordinate frame rotating with the field, gives an effective constant field applied along the X -axis plus the effective bias field in the Z -direction, $\hat{H}_\sigma'' = -l\hbar\omega\sigma_z - \frac{\Delta}{2}\sigma_x$. The first term is simply $-\omega \cdot \hbar\mathbf{l}$, that appears in the frame rotating at the mechanical angular velocity ω , projected into the $|\uparrow\rangle$ and $|\downarrow\rangle$ states. Real bias magnetic field B adds the term $-\mathbf{B} \cdot (\mu_B\mathbf{l})$ to the Hamiltonian. When the field is applied along the Z -axis the full two-state Hamiltonian in the rotating (SQUID) frame of reference becomes $\hat{H}_\sigma'' = -l(\hbar\omega + \mu_B B)\sigma_z - \frac{\Delta}{2}\sigma_x$. This proves that the rotation of a truncated two-state SQUID system satisfies Larmor theorem. It is equivalent to the magnetic field $B/\omega = \hbar/\mu_B \sim 10^{-7}\text{Oe/Hz}$. Effective fields generated by slow rotations of the equipment must have negligible effect on the flux qubit. However, the effect of local dynamic shear deformations on a microscopic SQUID must be noticeable because the corresponding angular velocities $(k v)(\Delta/\hbar)$ can easily reach 10^7Hz , providing effective fields in the range of $1G$.

Experiments with flux qubits have shown that significant decoherence comes from $1/f$ noise, the origin of which has been debated^{15,16}. Notice in this connection that relaxation of microscopic shear strains in a solid must be a source of dynamical local twists that, according to the above discussion, generate local effective magnetic fields. It is, therefore, plausible that relaxation of shear strains at the location of the qubit is, in fact, responsible

for the observed $1/f$ noise affecting quantum dynamics of the qubit.

Another observation worth mentioning is amplification of decoherence in a system of flux qubits positioned in close proximity to each other. This effect may be important in designing architectures of flux qubits if they are to be used for quantum computing. It will reveal itself when N microscopic qubits with identical tunnel splitting Δ are positioned within the wavelength of sound of frequency Δ/\hbar . As has been demonstrated in Section III B, radiation of sound by such a system and, thus, decoherence will be amplified by a factor N^2 . This is an acoustic analogue of Dicke superradiance that may impose an upper limit on the density of flux qubits. One way to avoid this effect in a dense assembly of qubits would be to use qubits of significantly different Δ .

In Section III C we studied renormalization of the tunnel splitting of a flux qubit arising from its interaction with zero-point shear deformations. The magnetic moment of the current of strength J in a loop of area a is $\mu = Ja/c$, which gives $l = Ja/(c\mu_B)$. With $a = \pi r^2$ and $k_{\max} = 2\pi/r$, the exponent in Eq. (84) becomes $\pi^4 \hbar J / (3c^2 \mu_B^2 \rho v_t)$. At $J \sim 1\mu\text{A}$ it is hopelessly small, thus, making this kind of renormalization irrelevant for a flux qubit. Notice in this connection that a similar effect, described by Eq. (84), may exist for the tunnel splitting of the atomic magnetic cluster. In this case l would be significantly smaller but k_{\max} would be much greater than for a flux qubit. An estimate for, e.g., a magnetic molecule frozen in solid He-4 shows that the exponent in Eq. (84) can easily be of order unity. As the He-solid approaches melting transition on decreasing pressure, its shear modulus would go to zero, resulting in the freezing of tunneling.

Finally, we would like to notice that treatment developed in this paper should apply to nanomechanical devices incorporating SQUIDs. Such devices have been recently made and measured^{17,18}. They open the whole new field of the entanglement of qubit states with mechanical oscillations. Possible manipulation of superconducting qubits by mechanical rotations is another interesting aspect of the research on nanomechanical superconducting qubits. Our model of a rigid rotator with a flux qubit may provide a framework for theoretical studies of these effects.

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